THE VELOCITY OF THE LOCK WATER LEVEL AT A LINEAR VARIATION OF THE FLOW IN THE FILLING (EMPTYING) CONDUIT

DUMITRU DINU[1], OVIDIU SORIN CUPŞA[2]

[1] Constanta Maritime University, dinud@imc.ro
[2] Constanta Maritime University,

ABSTRACT

The paper presents a mathematical model of the filling (emptying) of the lock chamber in a transition period, when the valves are opening or shutting. In this situation, we can consider a linear variation of the flow. We shall study the flow of the viscous fluid, no compressible, in an unsteady regime, through the lock camber filling (emptying) conduits with the help of the general equation of Navier-Stokes and the equation of continuity. To solve these equations we used the potential vector, which has a physical significance: the circulation of the potential vector on the perimeter of the conduit section is equal to the flow through this conduit. By approximation we established a hyperbolic variation of the velocity of the liquid level to an asymptote.

Key words: lock chamber, level velocity, linear variation of the flow, potential vector.

INTRODUCTION

An accuracy mathematical model of the filling (emptying) reservoir phenomena with liquid, in our case the lock chamber, presume the solving of the movement equations in the conditions very near to the reality (Fig. 1). We refer to the transition periods when the valves are opening or shutting, one to one or all together, and also to the fact that between the pool I and the lock chamber (between the lock chamber and the pool II)are the conduits, sometimes long enough, which influence the filling (emptying) process.

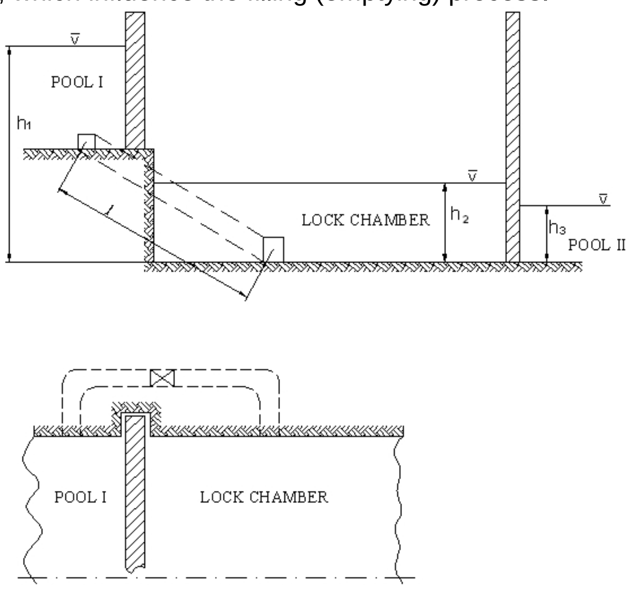

Fig. 1 The lock

The vector potential is a term taken from the electro-technique, in fact a symbol used to facilitate the mathematical calculation (for the solenoidal field $\nabla\bar{v}=0$, which involve $\bar{v}=rot\bar{A}$, $\bar{A}=\bar{A}(x,y.z.t)$ being the potential vector of the field if $\nabla\bar{A}=0$) apparent without physical significance. The physical significance of the potential vector was put into evidence in the paper [2]: the circulation of the potential vector on the perimeter of the conduit section is equal to the flow through this conduit. The potential vector help us to solve the Navier –Stokes equations to establish the velocity distribution in the conduit.

For the transitory regimes we can consider a linear variation of the flow. In this situation, we calculated the pressure gradient on the length of the filling (emptying) conduit of the lock chamber. Having this gradient, we could establish the variation of the hydrostatic pressure and the velocity of the lock chamber water level respectively.

1. THE USE OF POTENTIAL VECTOR IN THE STUDY OF FLUID MECHANICS

We shall study the flow of the viscous fluid, no compressible, in an unsteady regime, through the lock camber filling (emptying) conduits with the help of the general equation of Navier-Stokes and the equation of continuity.

In our theoretical study we shall use the transcription of these equations in cylindrical coordinates (r, θ, z). Oz being the axe of the conduit (Fig. 2):

$$\begin{aligned}
&\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}=\\
&=\upsilon\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\partial^2 v_r}{\partial z^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{v_r}{r^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial r};\\
&\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}=\\
&=\upsilon\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2}\right)-\frac{1}{\rho r}\frac{\partial p}{\partial\theta};\\
&\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}=\upsilon\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}\right)-\frac{1}{\rho}\frac{\partial p}{\partial z};\\
&\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}+\frac{v_r}{r}=0.
\end{aligned}\tag{1}$$

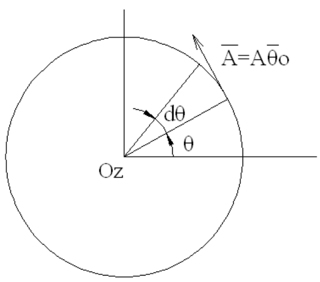

Fig. 2 The conduit in the cylindrival coordinates

The movement is axial-symmetric and the axe of the conduit coincides, as we already said, with the axe Oz. The components of the velocity will be:

$$v_r = v_\theta = 0, \qquad (2)$$
$$v_z = v(r,t).$$

As we know, the solenoidal (rotational) fields are characterized by $\nabla \bar{v} = 0$, which involve:

$$\bar{v} = rot\bar{A}, \qquad (3)$$

where $\bar{A}$ is a function of point and time, which represents the potential vector of the velocities field if $\nabla \bar{A} = 0$.

The potential vector of our movement will be:

$$\bar{A} = A(r,z)\overline{\theta_0} \qquad (4)$$

and it can be determinate with the relation:

$$\Delta\left(\frac{\partial \bar{A}}{\partial t} - \upsilon\Delta\bar{A}\right) = 0, \qquad (5)$$

a particularization for the axial-symmetric movement of the equation of the flow of real, no compressible fluids, written using the potential vector of the velocities field.

Solving the equation (5) using the Laplace transformation and taking into account that the circulation of the potential vector on the perimeter of the conduit section is equal to the flow through this conduit we shall obtain [ ]:

$$A(r,t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{st}\frac{Q^*}{2\pi r_0}\frac{2I_1(p) - pI_0(p_0)}{2I_1(p_0) - p_0 I_0(p_0)}ds, \qquad (6)$$

when I_0 and I_1 are the Bessel modify functions, rank 0, respectively 1, first sort, $p = r\sqrt{\frac{s}{\upsilon}}$ and $p_0 = r_0\sqrt{\frac{s}{\upsilon}}$.

Q^* is the Laplace transformation of the flow.

Knowing that the velocity is:

$$v = v_z = \left(rot\bar{A}\right)_z, \qquad (7)$$

we shall obtain the relation of the velocity:

$$v(r,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\frac{Q^*}{\pi r_0}\sqrt{\frac{s}{\upsilon}}\frac{I_0(p) - I_0(p_0)}{2I_1(p_0) - p_0 I_0(p_0)}ds. \qquad (8)$$

2. THE PRESSURE GRADIENT IN THE CASE OF LINIAR VARIATION OF THE FLOW

We consider a linear variation of the flow in the filling (emptying) conduit [1]. In this case, we propose to evaluate the variation of the level in the lock chamber.

So, the flow has a linear variation:

$$Q(t) = k_1 t + k_2. \qquad (9)$$

By applying Laplace transformation, we'll obtain:

$$Q^*(s)=\frac{k_1}{s^2}+\frac{k_2}{s}=\frac{k_1+k_2 s}{s^2}. \tag{10}$$

The expression of the potential vector becomes:

$$A(r,t)=\frac{1}{2\pi i}\frac{1}{2\pi r_0}\int_{b-i\infty}^{b+i\infty} e^{st}\,\frac{k_1+k_2 s}{s^2}\,\frac{2I_1(p)-pI_0(p_0)}{2I_1(p_0)-p_0I_0(p_0)}\,ds. \tag{11}$$

Solving the integrative, we'll have:

$$A(r,t)=2\pi i\frac{1}{2\pi i}\frac{1}{2\pi r_0}\sum \operatorname{Re} z=\frac{1}{2\pi r_0}\sum \operatorname{Re} z. \tag{12}$$

The poles of the function under integrative are:

s = 0, double pole and $s=-\frac{\upsilon}{r_0^2}\alpha_n^2\left(n\in N^*,\alpha\in R\right)$, where $\pm i\alpha_n=r_0\sqrt{\frac{s_n}{\upsilon}}$

represent the unnule sollutions of the equation:

$$2I_1(p_0)-p_0I_0(p_0)=0. \tag{13}$$

Knowing that:

$$I_k(w)=i^{-k}J_k(w) \tag{14}$$

we can write:

$$I_k(iw)=i^{-k}J_k(w) \tag{15}$$

and

$$I_0(iw)=J_0(w). \tag{16}$$

Replacing $w=p_0=i\alpha$ in equation (13), we'll have:

$$2J_1(\alpha)=J_0(\alpha), \tag{17}$$

$J_0(\alpha)$ and $J_1(\alpha)$ being the Bessel functions, rank 0, respectively 1, first sort.

Knowing that the no null solutions of this equation, $\pm\alpha_n$, can be determine graphically, we obtain:

$\alpha_1=5{,}15;\alpha_2=8{,}40;\alpha_3=11{,}61;\alpha_4=14{,}83$ etc.

The residuum of double pole, s=0, is given by the formula:

$$reziduu(0)=\frac{d}{ds}\left[s^2e^{st}\,\frac{k_1+k_2 s}{s^2}\,\frac{2I_1(p)-pI_0(P_0)}{2I_1(p_0)-p_0I_0(p_0)}\right]. \tag{18}$$

For simple pole $s=s_n=-\frac{\upsilon}{r_0^n}\alpha_n^2$ the residuum of the function under the integrative (11) will be:

$$reziduu(s_n)=s^{-\frac{\upsilon\alpha^2}{r_0^2}}\,\frac{k_1-k_2\frac{\upsilon\alpha_n^2}{r_0^2}}{\frac{\upsilon^2}{r_0^4}\alpha_n^4}\,\frac{2I_1\left(i\frac{r}{r_0}\alpha_n\right)-i\frac{r}{r_0}\alpha_nI_0(i\alpha_n)}{\frac{d}{ds}\left[2I_1(p_0)-p_0I_0(p_0)\right]_{s=s_n}}. \tag{19}$$

Finally we shall obtain the expression of the potential vector and velocity in the case of linear variation of the flow:

$$A(r,t)=\frac{1}{2\pi r_0}\left[(k_1t+k_2)\frac{r}{r_0}\left(2-\frac{r^2}{r_0^2}\right)-k_1\frac{r\left(r^2-r_0^2\right)^2}{24\upsilon r_0^3}-2\frac{r_0^2}{\upsilon}\sum_{n=1}^{m}e^{\frac{\upsilon}{r_0^2}\alpha_n^2t}\frac{k_1-k_2\frac{\upsilon}{r_0^2}\alpha_n^2}{\alpha_n^4}\frac{2J_1\left(\frac{r}{r_0}\alpha_n\right)-\frac{r}{r_0}\alpha_nJ_0(\alpha_n)}{J_1(\alpha_n)}\right];$$

(20)

$$v(r,t)=\frac{1}{\pi r_0}\left[(k_1t+k_2)\frac{1}{r_0}\left(1-\frac{r^2}{r_0^2}\right)-k_1\frac{r\left(r^2-r_0^2\right)^2\left(3r^2-r_0^2\right)}{48\upsilon r_0^3}-\frac{r_0^2}{\upsilon}\sum_{n=1}^{m}e^{-\frac{\upsilon}{r_0^2}\alpha_n^2t}\frac{k_1+k_2\frac{\upsilon}{r_0^2}\alpha_n^2}{\alpha_n^3}\frac{J_0\left(\frac{r}{r_0}\alpha_n\right)-\frac{r}{r_0}\alpha_nJ_0(\alpha_n)}{J_1(\alpha_n)}\right].$$

(21)

To verify we'll replace v(r,t) given by (21) in the flow formula:

$$Q(t)=\int_0^{r_0}2\pi r v(r,t)dr \tag{22}$$

and we'll obtain:

$$Q(t)=k_1t+k_2, \tag{23}$$

the beginning expression of linear flow.

To determine the pressure gradient, we shall use the Navier-Stokes equation, written in cylindrical coordinates, neglecting the masse forces (the third equation (1)) in which $v_r=v_\theta=0, v_z=v(r,t)$:

$$-\frac{\partial p}{\partial z}=\rho\frac{\partial v}{\partial t}-\eta\left(\frac{\partial^2 v}{\partial t^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right). \tag{24}$$

Taking into account that:

$$2J_1(\alpha)=J_0(\alpha); J_1^{'}(w)=\frac{J_1(w)}{w}+J_0(w); J_0^{'}(w)=-J_1(w); \tag{25}$$

will finally obtain the pressure gradient in the case of known linear variation of the flow:

$$-\frac{\partial p}{\partial z}=\frac{4\rho}{\pi r_0^2}\left[\frac{1}{3}k_1+\frac{2\upsilon}{r_0^2}(k_1t+k_2)-\sum_{n=1}^{m}e^{-\frac{\upsilon}{r_0^2}\alpha_n^2t}\frac{k_1-k_2\frac{\upsilon}{r_0^2}\alpha_n^2}{\alpha_n^2}\right]. \tag{26}$$

3. LIQUID LEVEL VARIATION

We have established the pressure gradient formula in the case of linear variation of the flow. Now we are interested in the variation of the lock chamber liquid level. We want to know the velocity of this level when the pumps fill the lock chamber. For this, denoting with f(t) the left part of the relation (26) we can write:

$$dp = -f(t)dz\,. \tag{27}$$

By integration the relation (27) on the length of the filling conduit, we shall obtain:

$$p_2 - p_1 = -f(t)(z_2 - z_1) \tag{28}$$

or

$$p_1 - p_2 = f(t)l\,, \tag{29}$$

l being the length of the conduit.

If p_1 is the constant hydrostatic pressure from the pool I and h_2 the level of the lock chamber liquid we'll have successively:

$$p_1 - \rho g h_2 = f(t)l\,;$$

$$\rho g h_2 = p_1 - f(t)l\,;$$

$$u(t) = \frac{h_2}{t} = \frac{1}{\rho g t}\left[p_1 - f(t)l\right]; \tag{30}$$

u(t) represents the rising velocity of the level.

By replacing f(t), we'll have:

$$u(t) = \frac{p_1}{\rho g t} - \frac{4l}{\pi g r_0^2}\left[\frac{1}{t}\left(\frac{k_1}{3} + \frac{2\upsilon k_2}{r_0^2}\right) + \frac{2\upsilon k_1}{r_0^2} - \sum_{n=1}^{m}\frac{1}{t}e^{-\frac{\upsilon}{r_0^2}\alpha_n^2}\,\frac{k_1 - k_2\dfrac{\upsilon}{r_0^2}\alpha_n^2}{\alpha_n^2}\right]. \tag{31}$$

We can easy demonstrate that the two last terms of the square bracket of the relation (31) – those under the sign of sum – are much more less that the first two:

$$\frac{\dfrac{k_1}{3t}}{\dfrac{k_1}{t\sum\limits_{n=1}^{m} e^{\frac{\upsilon}{r_0^2}\alpha_n^2 t}}} = \frac{\sum\limits_{n=1}^{m} e^{\frac{\upsilon}{r_0^2}\alpha_n^2 t}}{3} \rangle\rangle 1 \text{ and } \frac{\dfrac{2\upsilon k_2}{r_0^2 t}}{\dfrac{\upsilon k_2}{r_0^2 t\sum\limits_{n=1}^{m} e^{\frac{\upsilon}{r_0^2}\alpha_n^2 t}}} = 2\sum_{n=1}^{m} e^{\frac{\upsilon}{r_0^2}\alpha_n^2 t} \rangle\rangle 1\,,$$

because α_n has the over unit values (5.15; 8.4; 11.61; 14.83 etc.)

By approximation we can write:

$$u(t) \cong \frac{C_1}{t} + C_2\,, \tag{32}$$

where

$$C_1 = \frac{p_1}{\rho g} - \frac{4l}{\pi g r_0^2}\left(\frac{k_1}{3} + \frac{2\upsilon k_2}{r_0^2}\right) \text{ and } C_2 = \frac{8l\upsilon k_1}{\pi g r_0^4}\,,$$

a hyperbolic variation of the liquid level to the asymptote $u = \frac{8l\upsilon k_1}{\pi g r_0^4}$.

In the case of emptying of the lock chamber by free fall, admitting that the exit of the water is at atmospheric pressure, we'll have:

$$\rho g \frac{h_2 - h_3}{t} = \frac{f(t)}{t} l \,, \tag{33}$$

h_2 – the variable level of the lock chamber water, h_3 - the constant level of the pool II.

$$u(t) = \frac{h_2 - h_3}{t} = \frac{l}{\rho g t} f(t); \tag{34}$$

$$u(t) = \frac{4l}{\pi g r_0^2}\left[\frac{1}{t}\left(\frac{k_1}{3} + \frac{2\upsilon k_2}{r_0^2}\right) + \frac{2\upsilon k_1}{r_0^2} - \sum_{n=1}^{m} \frac{1}{t} e^{-\frac{\upsilon}{r_0^2}\alpha_n^2} \frac{k_1 - k_2 \frac{\upsilon}{r_0^2}\alpha_n^2}{\alpha_n^2}\right]. \tag{35}$$

We observe that:

$\lim_{t \to 0} u(t) = 0$ and $\lim_{t \to \infty} u(t) = \frac{8l\upsilon k_1}{\pi g r_0^4}$.

4. CONCLUSIONS

The mathematical model allows us to solve a quite delicate problem: the variation of the liquid level in the lock chamber in a transition period - the opening of a valve for example. The valve opening can be made thus the flow be linear. Solving Navier-Stokes equations using the potential vector, we have been able to establish the velocity variation in the conduit and the liquid level variation in the lock chamber.

The problem can be formulated conversely: taking a certain velocity, constant, of the liquid level, we'll determine the necessary flow for this level variation. In this situation the pressure gradient has a linear variation $\frac{\partial p}{\partial z} = at$, where a is a dimensional constant.

BIBLIOGRAPHY

1. **DINU D.** „ Considerations regarding the Flow through Circular Conduits in the case of Linear Time Variation of Pressure Gradient", The 5th International Scientific Conference on Naval Technologies TEHNONAV, Constanta, 2006.

2. **VASILESCU AL.A., PETREA F.** „Sur l'emploi du potential vectoriel du champ des vitesses dans la mécanique des fluides", Buletinul Universităţii din Galaţi, fascicol II, 1984.